Investment Horizon Effects

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Abstract: Boudry and Gray (2003) have documented that the optimal buy-and-hold demand for Australian stocks is not necessarily increasing in the investment horizon when returns are predictable. Such finding is in contrast with Barberis (2000) who shows that positive monotonic horizon effects predominate for US stocks. Using a closed-form approximation to the asset allocation problem, this paper relates the return dynamics to the investor's portfolio choice for different investment horizons. In the special case of a single risky asset, it is shown that return predictability under stationarity may induce both positive and negative horizon effects in the optimal allocation to the risky asset. The paper extends previous empirical results by solving for the optimal portfolio when two risky assets with predictable returns are available for investment.

Keywords: portfolio choice, return predictability, investment horizon effects

1. INTRODUCTION

Time diversification is based on the idea that above-average risky asset returns tend to be offset by below-average returns over long horizons if returns are not perfectly serially correlated. As a consequence, the variance of the annualized return diminishes with the number of years. Advocates of time diversification conclude that investors with longer horizons should hold a higher proportion of their portfolio in the risky asset. Such conclusion, however, has been subject to two main criticisms.[1] First, classic results by Samuelson (1969) and Merton (1969 and 1971) demonstrate that if the investor is free to rebalance his portfolio each period, he will choose a constant allocation to the risky asset regardless of the investment horizon, provided that he either has log utility preferences, or he has constant relative risk aversion and returns are independently and identically distributed (i.i.d.). Second, the variance of *total return* on the risky asset not only does not decrease, but actually increases with the investor's horizon for i.i.d. single-period returns. This is because although the probability of loss decreases with the investment horizon,

1 For recent reviews of the time diversification literature, see, for instance, Kritzman (1994), Bodie (1995), Connelly (1996), or Kritzman and Rich (1998).

the magnitude of the potential loss increases with time.[2] Long-term investments in stocks are therefore not safer than short-term investments for i.i.d. returns.

If returns are mean reverting, however, then the variance of total return increases, but it does so at a lower rate than in the i.i.d. case. This implies that stocks can be *relatively* less risky in the long run, which may lead investors to favor risky assets over long horizons. Kritzman and Rich (1998) address this issue in a simple and clarifying context. In particular, they study the problem of an investor who must choose between investing all his wealth in a safe asset and investing it in a risky asset. Under the assumption that the stock price process can be represented by a binomial tree with mean-reverting returns, they show numerically that a power utility investor with a constant relative risk aversion coefficient of two, will be indifferent between the risky asset and the safe asset in the short run, but will prefer the risky asset for a long-term investment. On the other hand, a longer investment horizon will lead the investor to choose the safe asset if the return distribution is mean averting, even if he was indifferent over a short horizon. Finally, for i.i.d. returns, the investor obtains the same expected utility from investing in the risky asset and the safe asset regardless of the investment horizon.

Barberis (2000) extends the analysis to a more complex setting. More specifically, he studies the effect of the investment horizon on the optimal portfolio choice of a power utility investor assuming a lognormal return generating model with a first-order autoregressive predictive variable[3] driving the time-varying conditional mean of log excess returns. Barberis (2000) also allows for a continuum of portfolio allocations between the risk-free and the risky asset. Using a numerical technique, Barberis (2000) shows that the optimal buy-and-hold investment in stocks is increasing in the investment horizon, independently of the initial value of the predictor. Barberis (2000) argues that predictability induces mean reversion in stock expected returns, which in turn makes the variance of cumulative returns grow less than linearly as the horizon increases. As a result, a risk averse long-term investor should allocate more to stocks than a short-term investor, consistently with Kritzman and Rich (1998).

The expected utility analysis of long-term portfolio choice under mean reversion, therefore, seems to have resurrected the idea of time diversification. This view, however, has been challenged by the empirical findings of Boudry and Gray (2003). Extending Barberis's (2000) approach to multiple predictors and Australian data, they find that some initial values of the predictors can cause the investor to allocate less to the risky asset as the investment horizon increases. Another source of discrepancy is the fact that optimal allocations are insensitive to the investment horizon in the long term. Finally, the effect of the investment horizon is not necessarily monotonic, as opposed to the US case.

Our limited understanding of investment horizon effects is perhaps explained by the lack of a closed-form solution to the static portfolio choice problem in the power utility case. In this paper, we shed light on the issue by using an analytical approximation to the investor's objective function that permits us to relate the return dynamics to the investor's optimal asset allocation for different horizons. The approximation has the traditional mean-variance form, except that risky asset returns

2 Time diversification is just an example of Samuelson's (1963) fallacy of large numbers, which he summarized with the following sentence (p. 54): 'it is not so much by adding new risks as by subdividing risks among more people that insurance companies reduce the risk of each.'
3 For evidence on return predictability, see, for instance, Campbell (1987), Campbell and Shiller (1988a and 1988b), Fama (1984 and 1990), or Fama and French (1988 and 1989).

are replaced by log returns and expected returns must be appropriately adjusted. In a portfolio choice model with two risky assets and multiple predictors, we then investigate how model parameters influence the evolution of the risk-return trade-off with the investment horizon. Interestingly, under the stationarity assumption, all investment horizon effects disappear in the long horizon limit. In the special case of a single risky asset, we show that under return predictability and negative serial correlation in asset returns, the horizon effect on the optimal demand for the risky asset can be either positive or negative. Two reasons explain this conclusion. First, changes in the conditional variance of cumulative log returns are not necessarily monotonic in the investment horizon: even if returns are negatively serially correlated, the asset's riskiness may grow less than linearly at short horizons and more than linearly at long horizons. Second, predictability implies that conditional expected cumulative returns may also grow less than linearly as the investment horizon grows to infinity. Intuitively, when the predictive variable is stationary, it tends to revert to its mean, so if it has a positive (negative) effect on expected returns and the initial value is above (below) its long-run mean, then future expected returns will tend to decrease. This decay in the mean may offset the positive horizon effect induced by lower relative risk. Empirical results for the US are therefore explained by a very low speed of mean reversion.

We illustrate the analysis empirically by computing the optimal demand of a buy-and-hold investor for long-term US Government bonds and stocks. While investment horizon effects are always positive when a stock portfolio is the only risky investable asset, the optimal demand for long-term bonds displays both positive and negative horizon effects depending on the initial value of the predictive variable and, in any case, disappear as the investment horizon grows to infinity. When both risky assets are simultaneously available for investment, we find that the investment horizon affects not only the optimal investment in the risky asset portfolio, but also the composition of the risky portfolio itself. Moreover, investment horizon effects have different signs for different initial values of the predictors. Finally, the paper shows how these results are changed when estimation risk is incorporated explicitly in the investor's problem.

To summarize, the paper's contribution to the literature is two-fold. First, it provides a theoretical analysis that relates the evolution of the risky asset return distribution over time to the optimal portfolio of a utility-maximizing investor. This analysis enables us to reconcile the empirical evidence on investment horizon effects under return predictability previously documented in the literature. Second, it extends Barberis's (2000) and Boudry and Gray's (2003) approach to the case when two risky assets are available for investment.

The rest of the paper is divided into the following sections: Section 2 presents the theoretical framework of the paper; Section 3 documents the empirical results; and finally, Section 4 concludes.

2. PORTFOLIO CHOICE UNDER RETURN PREDICTABILITY

In this section, we start by presenting the multiperiod portfolio problem of an investor with power utility who does not rebalance his portfolio. Throughout the paper, we follow Barberis (2000), but we consider two natural extensions of the original setting. First, we allow for multiple regressors in the predictive equation as in Boudry and Gray (2003), and, second, we include the possibility to invest in multiple risky assets. In the second part of the section, we investigate investment horizon effects for the special cases of two and one risky assets.

(i) The Investor's Problem

Consider the problem at time T of an investor who derives utility from his wealth at time $T + \hat{T}$. The investor does not rebalance his portfolio and can invest in $N + 1$ different assets: the one-month Treasury Bill with constant interest rate, r_f; and N risky assets. The continuously compounded return in excess of the riskless interest rate on the ith risky asset between periods $t - 1$ and t is denoted by r_{it}.

If initial wealth equals one and $\boldsymbol{\omega}' = [\omega_1 \; \omega_2 \ldots \omega_N]$ contains the fraction of the investor's initial wealth allocated to each risky asset, then the investor's terminal wealth is given by:

$$W_{T+\hat{T}} = (1 - \boldsymbol{\omega}'\boldsymbol{\iota}) \exp(r_f\hat{T}) + \sum_{i=1}^{N} \omega_i \exp(r_f\hat{T} + R_{i,T+\hat{T}}), \tag{1}$$

where $R_{i,T+\hat{T}}$ denotes the cumulative excess log return between T and $T + \hat{T}$, i.e., $R_{i, T+\hat{T}} = r_{i,T+1} + r_{i,T+2} + \ldots + r_{i,T+\hat{T}}$, and $\boldsymbol{\iota}$ is a vector of ones.

We assume that investor preferences can be represented by a utility function with constant relative risk aversion: $u(W) = \frac{W^{1-A}}{1-A}$, with $A > 1$. The problem can then be formally stated as follows:

$$\max_{\{\omega_i\}_{i=1}^{N}} \quad E_T\left\{\frac{\left[(1 - \boldsymbol{\omega}'\iota) \exp(r_f\hat{T}) + \sum_{i=1}^{N} \omega_i \exp(r_f\hat{T} + R_{i,T+\hat{T}})\right]^{1-A}}{1 - A}\right\} \tag{2}$$

We also assume the presence of short-sale and borrowing restrictions, so portfolio weights are constrained between zero and one.

Finally, excess returns are generated by the VAR model:

$$z_t = a + Bx_{t-1} + \epsilon_t, \tag{3}$$

where $z_t' = (r_{1t},\ldots,r_{Nt},x_t')$, $x_t = (x_{1t}, \ldots, x_{Mt})'$ and ϵ_t is identically and independently distributed according to $N(0, \Sigma)$. Vector x_t is the vector of M variables that predict expected returns. It is useful to rewrite the model as:

$$z_t = a + B_0 z_{t-1} + \epsilon_t,$$

where:

$$B_0 = [\mathbf{0}_{(N+M)\times N} \; B].$$

The sum $Z_{T+\hat{T}} = z_{T+1} + z_{T+2} + \ldots + z_{T+\hat{T}}$ is distributed as a multivariate normal with mean μ_{sum} and variance Σ_{sum} given by:

$$\begin{aligned}\mu_{sum} = &\hat{T}a + (\hat{T} - 1)B_0a + (\hat{T} - 2)B_0^2 a + \cdots + B_0^{\hat{T}-1}a \\ &+ (B_0 + B_0^2 + \cdots + B_0^{\hat{T}})z_T,\end{aligned} \tag{4}$$

$$\sum\nolimits_{sum} = \sum$$
$$+ (I + B_0)\Sigma(I + B_0)'$$
$$+ (I + B_0 + B_0^2) \sum (I + B_0 + B_0^2)' \tag{5}$$
$$\vdots$$
$$+ \left(I + B_0 + \cdots + B_0^{\hat{T}-1}\right)\Sigma\left(I + B_0 + \cdots + B_0^{\hat{T}-1}\right)',$$

Since a solution to (2) is not known in closed form, Barberis (2000) proposes to evaluate the objective function numerically by computing the integral:

$$\int u(W_{T+\hat{T}})p(\mathbf{R}_{T+\hat{T}}|\Phi_T)d\mathbf{R}_{T+\hat{T}},$$

where $\mathbf{R}_{T+\hat{T}}$ is the column vector of cumulative excess log returns, $p(\mathbf{R}_{T+\hat{T}}/\Phi_T)$ is the probability density function of the predictive distribution, and Φ_T is the investor's information set at T. The approach therefore involves estimating the parameters in (3) and then drawing realizations of $Z_{T+\hat{T}}$ from the multivariate normal distribution with mean and variance $\hat{\mu}_{sum}$ and $\hat{\Sigma}_{sum}$. Next, for every simulated realization, the value of the integral's argument is calculated, and finally the arithmetic mean of all values is obtained. The numerical solution to the optimization problem is the portfolio choice that maximizes the integral's numerical value.

Barberis (2000) also considers the possibility that the investor decides to take estimation risk into account. The predictive distribution then results from integrating the joint distribution of future returns and the parameter vector in (3), denoted by θ. Namely, the investor will obtain the probability density function for returns as:

$$p(\mathbf{R}_{T+\hat{T}}|\Phi_T) \equiv p(\mathbf{R}_{T+\hat{T}}|z) = \int p(\mathbf{R}_{T+\hat{T}}, \theta|z)d\theta$$
$$= \int p(\mathbf{R}_{T+\hat{T}}|\theta, z)p(\theta|z)d\theta,$$

where $z = z_1, z_2, \ldots, z_T$. The probability density $p(\mathbf{R}_{T+\hat{T}}|\theta, z)$ corresponds to a multivariate normal distribution with mean and variance equal to μ_{sum} and Σ_{sum}. An investor who takes parameter uncertainty into account therefore evaluates the following integral:

$$\int u(W_{T+\hat{T}})p(\mathbf{R}_{T+\hat{T}}|\theta, z)p(\theta|z)d\theta d\mathbf{R}_{T+\hat{T}}.$$

In practical terms, a realization of the vector θ is first drawn from the posterior distribution,[4] $p(\theta|z)$, and then a possible realization of $\mathbf{R}_{T+\hat{T}}$ is simulated using the distribution of future returns conditional on those parameter values. In the empirical application of Section 3, our choices for the size of the simulated sample are 100,000 in the case of no parameter uncertainty and 10,000 when estimation risk is taken into account.

4 See Barberis (2000) for details on the posterior distribution.

If parameter uncertainty is ignored, it is possible to obtain an analytical approximation to the investor's objective function. Such closed-form solution, albeit inexact, enables us to investigate the investment horizon effects documented by Barberis (2000) and Boudry and Gray (2003).

Since $A > 1$, we can restate (2) as:

$$\min_{\{\omega_i\}_{i=1}^N} \quad E_T\left\{\left[(1 - \boldsymbol{\omega}'\boldsymbol{\iota})\exp(r_f\hat{T}) + \sum_{i=1}^N \omega_i \exp(r_f\hat{T} + R_{i,T+\hat{T}})\right]^{1-A}\right\}$$

and, since wealth is always positive due to portfolio constraints, we can also write:

$$\min_{\{\omega_i\}_{i=1}^N} \quad \log E_T\left\{\left[(1 - \boldsymbol{\omega}'\boldsymbol{\iota})\exp(r_f\hat{T}) + \sum_{i=1}^N \omega_i \exp(r_f\hat{T} + R_{i,T+\hat{T}})\right]^{1-A}\right\}$$

Consider the following second-order Taylor approximation for log of wealth adapted from Campbell and Viceira (2002):

$$\log\left[(1 - \boldsymbol{\omega}'\boldsymbol{\iota})\exp(r_f\hat{T}) + \sum_{i=1}^N \omega_i \exp(r_f\hat{T} + R_{i,T+\hat{T}})\right]$$
$$\approx r_f\hat{T} + \boldsymbol{\omega}'\mathbf{R}_{T+\hat{T}} + \frac{1}{2}\boldsymbol{\omega}'\boldsymbol{v}_{\hat{T}}^2 - \frac{1}{2}\boldsymbol{\omega}'V_{\hat{T}}\boldsymbol{\omega}',$$

where $\boldsymbol{v}_{\hat{T}}^2$ is a vector containing the variances of the elements in $\mathbf{R}_{T+\hat{T}}$, and $V_{\hat{T}}$ is the variance-covariance matrix of $\mathbf{R}_{T+\hat{T}}$.

The approximation makes wealth lognormal and therefore enables us to write the objective function in mean-variance terms:

$$\log E_T W_{T+\hat{T}}^{1-A} \approx (1-A)E_T \log W_{T+\hat{T}} + \frac{1}{2}(1-A)^2 \text{var}_T \log W_{T+\hat{T}}.$$

Dividing by $(1 - A)$ the objective function becomes:

$$r_f\hat{T} + \boldsymbol{\omega}'\left(E_T\mathbf{R}_{T+\hat{T}} + \frac{1}{2}\boldsymbol{v}_{\hat{T}}^2\right) - \frac{A}{2}\boldsymbol{\omega}'V_{\hat{T}}\boldsymbol{\omega}'. \tag{6}$$

This expression has the traditional mean-variance form except that: (i) excess returns are replaced by log excess returns when computing both the mean and the variance-covariance terms, and (ii) mean log excess return of the i-th asset must be corrected by adding the term $\frac{1}{2}v_{i\hat{T}}^2 = \frac{1}{2}var_T(R_{i,T+\hat{T}})$. For example, in the case of two risky assets, the investor maximizes:

$$r_f\hat{T} + \omega_1\left(E_T R_{1,T+\hat{T}} + \frac{1}{2}\text{var}_T(R_{1,T+\hat{T}})\right) + \omega_2\left(E_T R_{2,T+\hat{T}} + \frac{1}{2}\text{var}_T(R_{2,T+\hat{T}})\right)$$
$$- \frac{A}{2}\left(\omega_1^2\text{var}_T(R_{1,T+\hat{T}}) + \omega_2^2\text{var}_T(R_{2,T+\hat{T}}) + 2\omega_1\omega_2\text{cov}_T(R_{1,T+\hat{T}}, R_{2,T+\hat{T}})\right)$$

subject to $0 \le \omega_i \le 1$, $i = 1, 2$.

In the particular case of a single risky investable asset, if $R_{T+\hat{T}}$ denotes the risky asset's cumulative log excess return, and ω the fraction of initial wealth invested in the risky asset, then the optimal portfolio is:

$$\omega^* = 0 \qquad\qquad \text{for } E_T R_{T+\hat{T}} + \frac{1}{2}\text{var}_T(R_{T+\hat{T}}) < 0,$$

$$\omega^* = 1 \qquad\qquad \text{for } E_T R_{T+\hat{T}} + \frac{1}{2}\text{var}_T(R_{T+\hat{T}}) > A\text{var}_T(R_{T+\hat{T}}) \qquad (7)$$

$$\omega^* = \frac{E_T R_{T+\hat{T}}}{A\text{var}_T(R_{T+\hat{T}})} + \frac{1}{2A} \quad \text{otherwise,}$$

since the objective function is globally concave in ω.

(ii) Investment Horizon Effects

In order to understand how the investment horizon changes the moments of cumulative log excess returns, we start by restricting our attention to the two risky asset case with multiple predictors, and then focus on the single risky asset, single predictor special case.

Assume that the two risky assets are a portfolio of stocks and a portfolio of bonds, denoted with subscripts s and b, respectively. For the sake of tractability, we assume also that variables predicting stock and bond returns follow univariate first order autoregressive processes, although we allow for correlation in the error terms across variables. Return dynamics are therefore governed by the following VAR model:

$$r_{st+1} = \alpha_s + \beta_{s1}x_{1t} + \cdots + \beta_{sM}x_{Mt} + \varepsilon_{st+1}$$
$$r_{bt+1} = \alpha_b + \beta_{b1}x_{1t} + \cdots + \beta_{bM}x_{Mt} + \varepsilon_{bt+1}$$
$$x_{1t+1} = \gamma_1 + \phi_1 x_{1t} + \varepsilon_{1t+1}$$
$$\vdots$$
$$x_{Mt+1} = \gamma_M + \phi_M x_{Mt} + \varepsilon_{Mt+1}$$

$$\begin{pmatrix} \varepsilon_{st+1} \\ \varepsilon_{bt+1} \\ \varepsilon_{1t+1} \\ \vdots \\ \varepsilon_{Mt+1} \end{pmatrix} \sim N \left[0, \begin{pmatrix} \sigma_s^2 & \sigma_{sb} & \sigma_{s1} & \cdots & \sigma_{sM} \\ & \sigma_b^2 & \sigma_{b1} & \cdots & \sigma_{bM} \\ & & \sigma_1^2 & & \sigma_{1M} \\ & & & \ddots & \vdots \\ & & & & \sigma_M^2 \end{pmatrix} \right] \qquad (8)$$

where we assume that $|\phi_k| < 1$, $\forall k$. Let us study stock returns, since conclusions are identical for either risky asset. The first-order stationary autocovariance of excess returns is given by:

$$\text{cov}(r_{st}, r_{st+1}) = \sum_{k=1}^{M} \beta_{sk}\sigma_{sk} + \sum_{k=1}^{M} \beta_{sk}\phi_k \sum_{j=1}^{M} \beta_{sj}\phi_j \frac{\sigma_{kj}}{1 - \phi_k\phi_j}.$$

If the predictive coefficient, β_{sk}, and the covariance between returns and the shock to the predictive variable, σ_{sk}, have opposite signs, then shocks to the predictive variable have opposite effects on the current period's return (through the contemporaneous covariance) and next period's expected return (through the predictive equation) contributing to negative serial correlation in stock returns. Similarly, negative values for $\beta_{sk}\beta_{sj}\phi_k\phi_j\sigma_{kj}$ tend to induce negative autocorrelation in the return series. Assume, for instance, that $\beta_{sk}, \beta_{sj}, \phi_k, \phi_j > 0$ and $\sigma_{kj} < 0$. In this case, a positive shock to x_{kt} increases next period's expected return but is likely to be contemporaneous with a negative shock to x_{jt}, which, in turn, decreases x_{jt+1} and therefore the expected value of r_{st+2}. Negative serial correlation in returns, however, does not guarantee that the relative risk-return trade-off for the risky asset improves with the investment horizon. To see this, we derive, next, the expressions for the moments of cumulative returns.

From (5) and (8) it follows that, under stationarity, the conditional variance of $R_{s,\ T+\hat{T}}$ evolves according to:

$$v_{s\hat{T}} \equiv \text{var}_T(R_{s,T+\hat{T}}) = v_{s\hat{T}-1} + \sigma_s^2 + 2\sum_{k=1}^{M}\beta_{sk}\frac{1-\phi_k^{\hat{T}-1}}{1-\phi_k}\sigma_{sk}$$

$$+ \sum_{k=1}^{M}\beta_{sk}\left(\frac{1-\phi_k^{\hat{T}-1}}{1-\phi_k}\right)\sum_{j=1}^{M}\beta_{sj}\left(\frac{1-\phi_j^{\hat{T}-1}}{1-\phi_j}\right)\sigma_{kj},$$

for $\hat{T} = 1, 2, \ldots, \infty$, with $v_{s0} = 0$.

Similarly, the cumulative mean of stock returns is given by:

$$m_{s\hat{T}} \equiv E_T(R_{s,\ T+\hat{T}}) = m_{s\hat{T}-1} + \alpha_s + \sum_{k=1}^{M}\beta_{sk}\left(\gamma_k\frac{1-\phi_k^{\hat{T}-1}}{1-\phi_k} + \phi_k^{\hat{T}-1}x_{kT}\right),$$

for $\hat{T} = 1, 2, \ldots, \infty$, with $m_{s0} = 0$.

To complete the analysis, the covariance of cumulative stock and bond returns evolves with $\hat{T}$ according to:

$$\text{cov}_{\hat{T}} \equiv \text{cov}_T(R_{s,\ T+\hat{T}}, R_{b,\ T+\hat{T}})$$

$$= \text{cov}_{\hat{T}-1} + \sigma_{sb} + \sum_{k=1}^{M}\beta_{sk}\frac{1-\phi_k^{\hat{T}-1}}{1-\phi_k}\sigma_{sk} + \sum_{k=1}^{M}\beta_{bk}\frac{1-\phi_k^{\hat{T}-1}}{1-\phi_k}\sigma_{bk}$$

$$+ \sum_{k=1}^{M}\beta_{sk}\left(\frac{1-\phi_k^{\hat{T}-1}}{1-\phi_k}\right)\sum_{j=1}^{M}\beta_{bj}\left(\frac{1-\phi_j^{\hat{T}-1}}{1-\phi_j}\right)\sigma_{kj}$$

for $\hat{T} = 1, 2, \ldots, \infty$, with $\text{cov}_0 = 0$.

Note that, if excess log returns were i.i.d. ($\beta_{ik} = 0, i = s, b, \forall k$), both the variance-covariance matrix and the mean vector in (6) would be proportional to $\hat{T}$, so the investment horizon could be removed from the objective function and the optimal risky asset portfolio would have the same composition regardless of the investment horizon. In the non-i.i.d. case, however, predictability in risky asset returns will cause the whole mean-variance frontier to change with the investor's horizon, which

will induce investment horizon effects both in the optimal allocation to the risky portfolio as well as in the composition of the risky portfolio itself.

More specifically, $v_{s\hat{T}}$ may grow more or less than linearly as the investor's horizon increases, depending on whether $v_{\hat{T}} - v_{\hat{T}-1}$ is increasing in $\hat{T}$. For the simple case of two predictive variables ($M = 2$) and non-negative ϕ_1, ϕ_2, changes in the conditional cumulative variance will decrease with the investment horizon if:

$$\frac{\partial(v_{s\hat{T}} - v_{s\hat{T}-1})}{\partial \hat{T}} = 2\beta_{s1}\sigma_{s1}\phi_1^{\hat{T}-1}\frac{\ln\phi_1}{-1+\phi_1} + 2\beta_{s2}\sigma_{s2}\phi_2^{\hat{T}-1}\frac{\ln\phi_2}{-1+\phi_2}$$

$$+ 2\beta_{s1}^2\sigma_1^2\frac{1-\phi_1^{\hat{T}-1}}{1-\phi_1}\phi_1^{\hat{T}-1}\frac{\ln\phi_1}{-1+\phi_1} + 2\beta_{s2}^2\sigma_2^2\frac{1-\phi_2^{\hat{T}-1}}{1-\phi_2}\phi_2^{\hat{T}-1}\frac{\ln\phi_2}{-1+\phi_2}$$

$$+ 2\beta_{s1}\beta_{s2}\sigma_{12}\left(\frac{1-\phi_2^{\hat{T}-1}}{1-\phi_2}\phi_1^{\hat{T}-1}\frac{\ln\phi_1}{-1+\phi_1} + \frac{1-\phi_1^{\hat{T}-1}}{1-\phi_1}\phi_2^{\hat{T}-1}\frac{\ln\phi_2}{-1+\phi_2}\right) \tag{9}$$

$$< 0.$$

It thus follows that for non-negative ϕ_k, if β_{sk} and σ_{sk}, have opposite signs, the k-th predictive variable not only may induce negative serial correlation in the risky asset returns, but it also tends to reduce the relative riskiness of a long-term investment in the risky asset as long as:

$$\beta_{sk}\sigma_{sk} < \beta_{sk}^2\sigma_k^2\frac{1-\phi_k^{\hat{T}-1}}{1-\phi_k}.$$

Also, when the effects of the predictive variables on the risky asset's expected return, i.e., β_{s1} and β_{s2}, have the same (opposite) sign, and the contemporaneous covariance between both variables is positive (negative) the last term in the left hand side of (9) is negative, which contributes towards making the conditional variance of the risky asset's return grow less than linearly with the investor's horizon.

The cumulative mean, on the other hand, grows less than linearly with the investment horizon if:

$$\frac{\partial(m_{s\hat{T}} - m_{s\hat{T}-1})}{\partial \hat{T}} = \sum_{k=1}^{M}\beta_{sk}\phi_k^{\hat{T}-1}\frac{\ln\phi_k}{-1+\phi_k}(\gamma_k - x_{kT}(1-\phi_k)) < 0. \tag{10}$$

Boudry and Gray (2003) perform their comparative statics exercises by setting the initial value of one predictive variable below or above its stationary mean, $\frac{\gamma_k}{1-\phi_k}$, while leaving the rest of predictors initially at their long-run means. It is clear from (10) that when the value of the perturbed variable is initially below (above) its stationary mean and $\beta_{sk} > 0$ (< 0), the mean of cumulative returns will grow more than linearly with the investment horizon.

Finally, the investment horizon also affects the composition of the risky asset portfolio as well as its total riskiness relative to that of the risk-free asset through the covariance of risky asset returns. In particular, changes in the conditional

covariance will decrease with the investment horizon if:

$$\frac{\partial(\text{cov}_{\hat{T}} - \text{cov}_{\hat{T}-1})}{\partial \hat{T}} = \beta_{s1}\sigma_{s1}\phi_1^{\hat{T}-1}\frac{\ln\phi_1}{-1+\phi_1} + \beta_{s2}\sigma_{s2}\phi_2^{\hat{T}-1}\frac{\ln\phi_2}{-1+\phi_2}$$

$$+ \beta_{b1}\sigma_{b1}\phi_1^{\hat{T}-1}\frac{\ln\phi_1}{-1+\phi_1} + \beta_{b2}\sigma_{b2}\phi_2^{\hat{T}-1}\frac{\ln\phi_2}{-1+\phi_2}$$

$$+ 2\beta_{s1}\beta_{b1}\sigma_1^2\frac{1-\phi_1^{\hat{T}-1}}{1-\phi_1}\phi_1^{\hat{T}-1}\frac{\ln\phi_1}{-1+\phi_1} + 2\beta_{s2}\beta_{b2}\sigma_2^2\frac{1-\phi_2^{\hat{T}-1}}{1-\phi_2}\phi_2^{\hat{T}-1}\frac{\ln\phi_2}{-1+\phi_2} \qquad (11)$$

$$+ (\beta_{s1}\beta_{b2} + \beta_{s2}\beta_{b1})\sigma_{12}\left(\frac{1-\phi_2^{\hat{T}-1}}{1-\phi_2}\phi_1^{\hat{T}-1}\frac{\ln\phi_1}{-1+\phi_1} + \frac{1-\phi_1^{\hat{T}-1}}{1-\phi_1}\phi_2^{\hat{T}-1}\frac{\ln\phi_2}{-1+\phi_2}\right)$$

$$< 0.$$

Clearly, under the stationarity assumption, in the long horizon limit ($\hat{T}\to\infty$), the expressions in (9), (10) and (11) all converge to zero, which implies that the optimal portfolio becomes independent of the investment horizon for very large $\hat{T}$.

In order to reconcile the empirical results obtained by Barberis (2000) and Boudry and Gray (2003), we focus, next, on the single risky asset, single predictor case. It follows from (7) that positive horizon effects on the optimal unconstrained allocation to the risky asset are a consequence of the variance of cumulative log returns growing slower than mean cumulative log returns with $\hat{T}$.

The return generating model in the single risky asset, single predictor case becomes:

$$r_{t+1} = \alpha + \beta x_t + \varepsilon_{st+1}$$
$$x_{t+1} = \gamma + \phi x_t + \varepsilon_{1t+1}$$
$$\begin{pmatrix} \varepsilon_{st+1} \\ \varepsilon_{1t+1} \end{pmatrix} \sim N\left[0, \begin{pmatrix} \sigma_s^2 & \sigma_{s1} \\ \sigma_{s1} & \sigma_1^2 \end{pmatrix}\right]. \qquad (12)$$

Under the stationarity assumption, i.e., $|\phi| < 1$, the first-order autocovariance of excess returns is given by:

$$\text{cov}(r_t, r_{t+1}) = \beta\sigma_{s1} + \frac{\beta^2\sigma_1^2\phi}{1-\phi^2}$$

so returns are serially negatively correlated in the long run if $\beta\sigma_{s1} < -\beta^2\sigma_1^2\frac{\phi}{1-\phi^2}$.

The variance of the cumulative return between periods T and $T + \hat{T}$ evolves with $\hat{T}$ according to:

$$v_{\hat{T}} \equiv \text{var}_T(R_{T+\hat{T}}) = v_{\hat{T}-1} + \sigma_s^2 + 2\beta\frac{1-\phi^{\hat{T}-1}}{1-\phi}\sigma_{s1} + \beta^2\left(\frac{1-\phi^{\hat{T}-1}}{1-\phi}\right)^2\sigma_1^2, \qquad (13)$$

which holds for $\hat{T} = 1, 2, \ldots, \infty$, with $v_0 = 0$. For non-negative ϕ, one-period changes in the variance of cumulative returns are decreasing in the horizon if:

$$\frac{\partial(v_{\hat{T}} - v_{\hat{T}-1})}{\partial \hat{T}} = 2\beta\sigma_{s1}\phi^{\hat{T}-1}\frac{\ln\phi}{-1+\phi} + 2\beta^2\sigma_1^2\frac{1-\phi^{\hat{T}-1}}{1-\phi}\phi^{\hat{T}-1}\frac{\ln\phi}{-1+\phi} \quad (14)$$
$$< 0,$$

so the variance grows less than linearly in $\hat{T}$, for $1 < \hat{T} < \infty$, if the following condition holds:

$$\beta\sigma_{s1} < -\beta^2\sigma_1^2\frac{1-\phi^{\hat{T}-1}}{1-\phi}. \quad (15)$$

Note that, even if β and σ_{s1} have opposite signs, it is not possible to assert whether their joint effect is enough to cause the variance to always grow less than linearly. With negatively serially correlated returns, $v_{\hat{T}}$ could be concave in $\hat{T}$ at short horizons but convex at longer horizons.

As for the mean of cumulative returns:

$$m_{\hat{T}} \equiv E_T(R_{T+\hat{T}}) = m_{\hat{T}-1} + \alpha + \beta\gamma\frac{1-\phi^{\hat{T}-1}}{1-\phi} + \beta\phi^{\hat{T}-1}x_T, \quad (16)$$

for $\hat{T} = 1, 2, \ldots, \infty$, with $m_0 = 0$. The mean increases in $\hat{T}$ only if $\alpha + \beta\gamma\frac{1-\phi^{\hat{T}-1}}{1-\phi} + \beta\phi^{\hat{T}-1}x_T > 0$. So, even if the variance of cumulative returns grows less than linearly in the investment horizon, a diminishing mean might cause the optimal allocation to decrease for long horizons. Moreover, assuming that the mean is positive and increasing in $\hat{T}$, it could grow slower than the variance. To see this, note that differentiating $m_{\hat{T}} - m_{\hat{T}-1}$ with respect to $\hat{T}$ gives:

$$\frac{\partial(m_{\hat{T}} - m_{\hat{T}-1})}{\partial \hat{T}} = \beta\phi^{\hat{T}-1}\frac{\ln\phi}{-1+\phi}(\gamma - x_T(1-\phi)).$$

So, for $\hat{T} < \infty$, the mean of cumulative returns $m_{\hat{T}}$ grows less than linearly with $\hat{T}$ if the predictor's initial value is above (below) its long-run mean when β is positive (negative), and more than linearly otherwise. Note that the effect of the predictor's initial value on $\frac{\partial(m_{\hat{T}}-m_{\hat{T}-1})}{\partial \hat{T}}$ has an opposite sign to that on $m_{\hat{T}}$. As a consequence, when x_T affects $E_T(r_{T+\hat{T}})$ positively, its increments decrease as the horizon becomes longer if the predictor is initially above its long-run mean. Intuitively, when the predictive variable is stationary, the influence of the predictor's initial value on each period's return decays with time. The larger the initial effect of the predicting variable, the larger the *loss* in the cumulative expected return as $\hat{T}$ increases.

Finally, under stationarity, the unconstrained optimal allocation to the risky asset converges to a constant:

$$\omega^* = \frac{\alpha + \dfrac{\beta\gamma}{1-\phi}}{A\left(\sigma_s^2 + \dfrac{2\beta\sigma s_1}{1-\phi} + \left(\dfrac{\beta}{1-\phi}\right)^2\sigma_1^2\right)} + \frac{1}{2A}. \quad (17)$$

Convergence, however, is far from obvious in Barberis's (2000) results for US stocks because of the high persistence in the predictor ($\phi = 0.9774$), but becomes evident in Boudry and Gray (2003), where ϕ ranges from 0.9039 to 0.9191.

To conclude this section, note that if returns are unpredictable, i.e. $\beta = 0$, then $v_{\hat{T}} = \hat{T}\sigma_1^2$ and $m_{\hat{T}} = \hat{T}\alpha$, and the unconstrained optimal allocation becomes:

$$\omega^* = \frac{\alpha}{A\sigma_s^2} + \frac{1}{2A}$$

regardless of the investment horizon.

3. EMPIRICAL RESULTS

In this section, we analyze empirically the effect of the investor's time horizon on his optimal asset allocation. We start by considering the case when only a single risky asset is available for investment and only one variable has predictive power over the risky asset return, and then extend the analysis to the more general model with two risky assets and multiple predictors.

As the single risky asset we consider, alternatively, a long-term US Government bond portfolio and a portfolio of US stocks, whose returns are proxied by the return on a representative constant maturity portfolio of 10-year US Government Bonds and on the value-weighted index of the New York Stock Exchange, respectively. Data were collected with monthly frequency for the period covering January 1959 to December 1998 from the CRSP (Center for Research in Security Prices) database. In order to compute excess returns, we also employ the one-month Treasury Bill rate series. When the constant risk-free rate is needed, we take r_f to be 0.0036, or 4.32% in annual terms.

Two predictive variables are selected for predicting stock and bond returns, respectively: the dividend yield on the stock index (DP), computed as the sum of dividends payed on the index over the previous 12 months divided by the current index level; and the trend of the stock index (*Trend*), defined as the difference between the logarithm of the index's current level and the average level over the previous 12 months.

We consider first the case of an investor who estimates the parameters in (12) and takes as the predictive distribution of returns a multivariate normal distribution with mean and variance $\hat{\mu}_{sum}$ and $\hat{\Sigma}_{sum}$, respectively. Values of the OLS parameter estimates in both cases are shown in Table 1. Figure 1 shows graphically the optimal allocation to stocks. The first conclusion we draw from Figure 1 is that the approximate solution performs remarkably well in this specific setting. Second, the investor allocates a larger fraction of his wealth to stocks the higher the value of DP and the longer the investment horizon. This result is consistent with Brennan, Schwartz and Lagnado (1997), Barberis (2000) or Kritzman and Rich (1998). In fact, it is to be expected from the results of the previous section for two reasons. First, condition (15) together with estimated parameters implies that the variance of cumulative returns grows less than linearly for any finite investment horizon. Second, the high estimated value of ϕ, close to unity, prevents conditional expected returns from reverting fast to the mean. As a consequence, the horizon effect on the variance

Table 1

Parameter Values in the Model with a Single Risky Asset and
One Predictor

Panel A

	Constant	DP
Stocks	−0.0044	0.0025
DP	0.0477	0.9862

	Constant	Trend
10-year Bonds	0.0025	−0.0007
Trend	0.2121	0.8787

Panel B

	Stocks	DP
Stocks	0.0019	−0.9296
DP		0.0290

	10-year Bonds	Trend
10-year Bonds	0.0005	0.2690
Trend		3.5932

Notes:
This table shows OLS estimates from the VAR model below. Panel A displays
regression coefficients while Panel B shows the variance-covariance matrix of
the error terms. r_t denotes the continuously compounded monthly excess
return on the risky asset (alternatively, US stocks or 10-year US
Government bonds) and x_t is the single predictor: the dividend yield (DP) or
the trend of the stock index (*Trend*). In Panel B, values above the main
diagonal are correlation coefficients.

$$r_{t+1} = \alpha + \beta x_t + \varepsilon_{1t+1}$$
$$x_{t+1} = \gamma + \phi x_t + \varepsilon_{2t+1}$$
$$\begin{pmatrix} \varepsilon_{1t+1} \\ \varepsilon_{2t+1} \end{pmatrix} \sim N \left[0, \begin{pmatrix} \sigma_s^2 & \sigma_{s1} \\ \sigma_{s1} & \sigma_1^2 \end{pmatrix} \right]$$

dominates that on the mean, and the optimal allocation to stocks is increasing for all
investment horizons and all initial values of x considered.

The case of long-term bonds is quite different even though the unconditional
autocovariance of excess returns is also negative. In Figure 2 we plot the investor's
optimal allocation to long-term US Government bonds for different investment
horizons and three different cases that correspond to three different initial values
of *Trend*. A pattern very similar to that documented by Boudry and Gray (2003) is
found: since the coefficient in the regression equation for *Trend* is negative, the
optimal demand is smaller but increasing in the investment horizon for an initial
value above the predictor's sample mean, and larger but decreasing for a lower
initial value. In particular, an investor with a relative risk aversion coefficient of 5
doubles his allocation to bonds when his investment horizon increases from 2.5
years to 10 years if the trend's initial value is high. These results are therefore
consistent with mean-driven horizon effects dominating variance-driven effects.

Figure 1

Optimal Allocation to Stocks
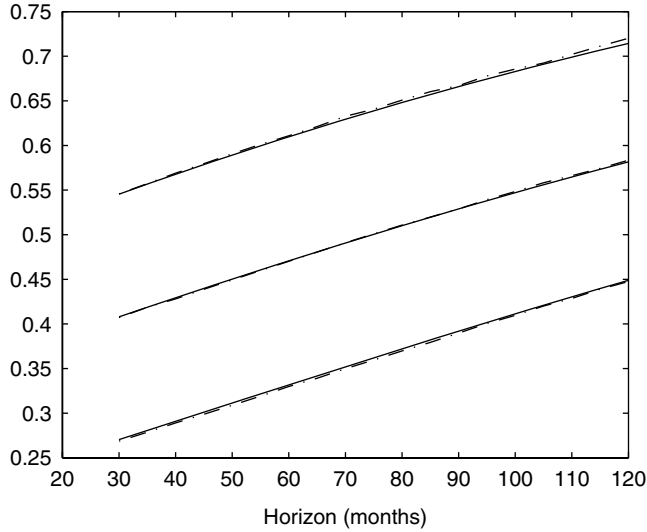
Notes:
This figure shows the optimal allocation to stocks as a function of the investment horizon when dividend yield (DP) is the only predictor, using the approximate analytical solution (solid lines) and the exact numerical solution (dash-dot lines). The bottom line corresponds to an initial value of DP equal to 3, the middle line to 4, and the top line to 5. The risk aversion coefficient is 10.

Next, we solve for the investor's portfolio choice when both risky assets are available for investment. In this case, we include two additional predictive variables:[5]

1. The default spread, proxied by the yield difference between Moody's Baa and Aaa rated corporate bonds.
2. The term spread, proxied by the yield difference between 10-year and one-year US Government Bonds.

Both series were obtained from the Internet site maintained by the US Federal Reserve.

If the investor performed an OLS regression of monthly excess stock and bond returns on the predictive variables in the previous month, he would obtain the following estimated coefficients and their associated p-values (in parentheses):

$$
\begin{aligned}
r_{st+1} = \ & -0.0077 + 0.0066 \ x_{1t} \ + 0.0007 \ x_{2t} \\
& (0.2181) \ (0.1608) \qquad (0.4257) \\
& \qquad\quad + 0.0054 \ x_{3t} \ - 0.0004 \ x_{4t} \ + \ \epsilon_{st+1} \\
& \qquad\quad (0.0071) \qquad (0.2291)
\end{aligned}
\tag{18}
$$

5 These predictors have been previously used by Fama and French (1988 and 1989) and Keim and Stambaugh (1986).

Figure 2

Optimal Allocation to Long-term US Government Bonds
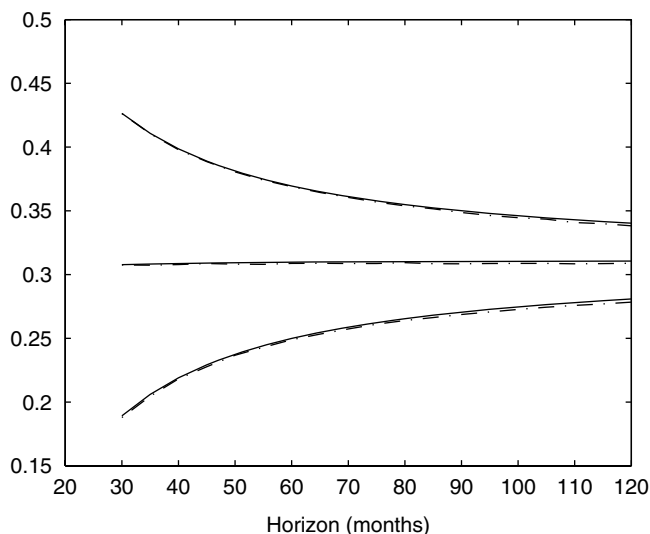
Notes:
This figure shows the optimal allocation to long-term US Government bonds as a function of the investment horizon when the trend of the stock index (*Trend*) is the only predictor, using the approximate analytical solution (solid lines) and the exact numerical solution (dash-dot lines). The top line corresponds to an initial value of *Trend* equal to −1, the middle line corresponds to 2, and the bottom line to 5. The risk aversion coefficient is 10.

$$
\begin{aligned}
r_{bt+1} = 0.0002 &\quad + 0.0038 \ x_{1t} \ 0.0008 \ x_{2t} \\
(0.4814) &\ (0.1333) \qquad (0.3288) \\
&\quad + 0.0024 \ x_{3t} - 0.0009 \ x_{4t} \quad + \quad \epsilon_{bt+1} \\
&\quad (0.0140) \qquad (0.0010)
\end{aligned}
\tag{19}
$$

where r_{st} and r_{bt} stand for continuously compounded excess returns on the stock and the 10-year bond indices respectively, and x_{it}, $i = 1, 2, 3,$ and 4 denote the value in month t of the default spread, DP, the term spread, and the stock index trend, respectively.

In regression (18) only the term spread is significantly different from zero, whereas in regression (19) the term spread and especially the trend have a significant predictive power over next month's expected return. On the other hand, the R^2 coefficient equals 0.0211 in the case of stocks and 0.0319 for bonds. These results show the magnitude of the investor's problem of deciding whether he should consider the empirical evidence on predictability or whether he should think that evidence is too weak to take the regression estimated coefficients as the true model parameter values. The Bayesian approach can thus be seen as an intermediate solution between both extreme attitudes. Next, we solve for the optimal portfolio choice of an investor who ignores estimation risk and an investor who explicitly

Table 2

Parameter Values in the Model with Two Risky Assets and Four Predictors

Panel A

	Constant	Def. Spread	DP	Term Spread	Trend
Stocks	−0.0076	0.0066	0.0007	0.0053	−0.0004
10-year Bonds	0.0003	0.0038	−0.0008	0.0024	−0.0009
Def. spread	−0.0039	0.9499	0.0208	−0.0142	−0.0047
DP	0.0475	−0.0462	1.0014	−0.0195	0.0022
Term spread	0.0452	0.1522	−0.0416	0.9581	−0.0126
Trend	−0.5764	0.3902	0.0791	0.1989	0.8705

Panel B

	Stocks	Bonds	Def. Spread	DP	Term Spread	Trend
Stocks	0.0019	0.2695	0.0783	−0.9379	0.1399	0.9903
Bonds		0.0005	0.2485	−0.2827	0.1620	0.2624
Def. spread			0.0102	−0.0990	0.1634	0.0723
DP				0.0289	−0.1935	−0.9325
Term spread					0.0704	0.1318
Trend						3.5845

Notes:

When the investor does not take parameter uncertainty into account, he takes the parameter values in the VAR model below as known. More specifically, parameter values in this table are the means from the parameter posterior distribution under a diffuse prior. Panel A displays regression coefficients while Panel B shows the variance-covariance matrix of the error terms. r_{st} denotes the continuously compounded monthly excess return on US stocks, r_{bt} denotes the continuously compounded monthly excess return on 10-year US Government bonds, $x_{1t}, x_{2t}, x_{3t}, x_{4t}$, are the default spread, the dividend yield (DP), the term spread, and the trend of the stock index (*Trend*). In Panel B, values above the main diagonal are correlation coefficients.

$$r_{st+1} = \alpha_s + \beta_{s1}x_{1t} + \beta_{s2}x_{2t} + \beta_{s3}x_{3t} + \beta_{s4}x_{4t} + \varepsilon_{st+1}$$
$$r_{bt+1} = \alpha_b + \beta_{b1}x_{1t} + \beta_{b2}x_{2t} + \beta_{b3}x_{3t} + \beta_{b4}x_{4t} + \varepsilon_{bt+1}$$
$$x_{1t+1} = \gamma_1 + \phi_{11}x_{1t} + \phi_{12}x_{2t} + \phi_{13}x_{3t} + \phi_{14}x_{4t} + \varepsilon_{1t+1}$$
$$x_{2t+1} = \gamma_2 + \phi_{21}x_{1t} + \phi_{22}x_{2t} + \phi_{23}x_{3t} + \phi_{24}x_{4t} + \varepsilon_{2t+1}$$
$$x_{3t+1} = \gamma_3 + \phi_{31}x_{1t} + \phi_{32}x_{2t} + \phi_{33}x_{3t} + \phi_{34}x_{4t} + \varepsilon_{3t+1}$$
$$x_{4t+1} = \gamma_4 + \phi_{41}x_{1t} + \phi_{42}x_{2t} + \phi_{43}x_{3t} + \phi_{44}x_{4t} + \varepsilon_{4t+1}$$

$$\begin{pmatrix} \varepsilon_{st+1} \\ \varepsilon_{bt+1} \\ \varepsilon_{1t+1} \\ \varepsilon_{2t+1} \\ \varepsilon_{3t+1} \\ \varepsilon_{4t+1} \end{pmatrix} \sim N \left[0, \begin{pmatrix} \sigma_s^2 & \sigma_{sb} & \sigma_{s1} & \sigma_{s2} & \sigma_{s3} & \sigma_{s4} \\ & \sigma_b^2 & \sigma_{b1} & \sigma_{b2} & \sigma_{b3} & \sigma_{b4} \\ & & \sigma_1^2 & \sigma_{12} & \sigma_{13} & \sigma_{14} \\ & & & \sigma_2^2 & \sigma_{23} & \sigma_{24} \\ & & & & \sigma_3^2 & \sigma_{34} \\ & & & & & \sigma_4^2 \end{pmatrix} \right]$$

incorporates estimation risk in his portfolio decision problem. When the investor does not recognize estimation risk, it will be assumed that he takes as the true parameter values the means of model parameters from the posterior distribution under a diffuse prior. These values are shown in Table 2.

Figure 3 summarizes the results graphically when all predictors' starting values coincide with their sample means. In the case of no parameter uncertainty, a long

Figure 3

Optimal Portfolio Choice as a Function of the Investment Horizon
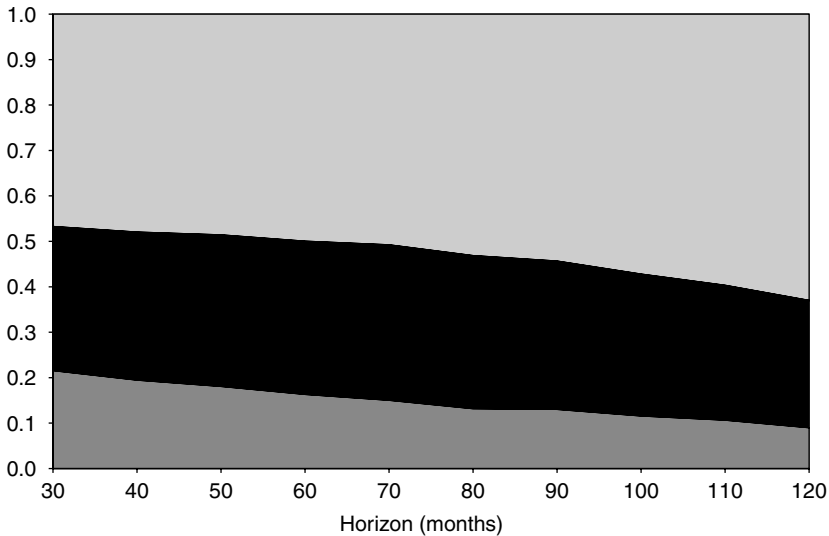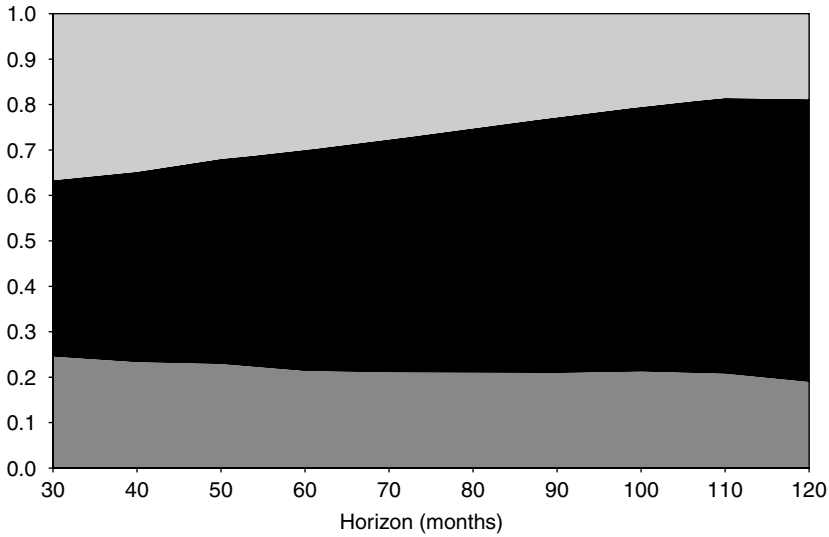
Notes:
The graph plots the static allocation to the riskless asset (light shade), 10-year bonds (medium shade) and stocks (dark shade) when returns are assumed to be generated by a VAR model with four predictors. The initial values of all predictive variables are set at their sample means. In the lower graph, the investor takes parameter uncertainty into account. The risk aversion coefficient is 10.

term investor will invest a higher fraction of his risky asset portfolio in stocks–and, therefore, a lower fraction in bonds–than a short-term investor. This consequence of predictability is interesting because the optimal allocation to bonds is independent of the investor's horizon when bonds are the only risky investable asset. Clearly, the sharp decline in the relative riskiness of equity as the investment

horizon increases tilts the optimal risky portfolio towards a smaller investment in the other risky asset: long term bonds. As a consequence, long term bonds account for almost 40% of a 30-month investor's risky portfolio, and only 23% in the case of a 10-year investor. This substitution effect, however, is almost completely compensated by a larger overall investment in the risky asset portfolio, whose attractiveness – relative to the safe investment – increases with the investment horizon: allocation to bonds declines by only 5% of the total portfolio as the horizon grows from 30 to 120 months. When estimation risk is considered, the decline in the relative riskiness of stocks is offset by the larger effect of parameter uncertainty at long horizons, so the optimal allocation to stocks remains almost constant. Long-term bonds, however, become relatively more risky for long investment horizons so the long-term investor replaces bonds with the risk-free asset.

Note that investment horizon effects in Figure 3 are solely driven by changes in the variance-covariance matrix, since the horizon is likely to have a negligible effect on the conditional mean vector according to (10). In Figures 4 to 7 we set the initial value of one of the predictors above or below its sample mean.

Graphs in Figure 4 correspond to an initial value of *Trend* equal to its 25th sample percentile for the cases of parameter certainty and parameter uncertainty. Changes in the trend's initial value affect expected returns on stocks and bonds in the same direction: a low current value of the variable raises next period's conditional mean returns on both assets. Consequently, a short-term investor allocates more to the risky asset portfolio than in the previous case. As the investment horizon increases, however, expected single-period returns decline due to mean reversion. As a consequence, the risk-return trade-off deteriorates for long-term bonds while stocks become a more interesting investment alternative due to their lower relative risk, so the weight of bonds in the optimal risky portfolio sharply decreases from 46% to 26%, while total investment in the risky portfolio increases due to the stronger appeal of stocks. Estimation risk, however, severely penalizes the long-term investor's position in risky assets, and induces a negative horizon effect on the optimal allocation both to bonds and stocks. This finding is consistent both with Barberis (2000) and Boudry and Gray (2003).

Figure 5 shows the results corresponding to the case when *Trend*'s initial value is set at its 75th sample percentile. Although the predictor's effect on expected bond returns suggests that long term bonds are a better investment at long horizons, the optimal fraction of bonds in the risky asset portfolio remains almost the same at both short and long horizons. Given the improvement in the risky investment opportunity set at long horizons, investment in the risk-free asset reduces from 60% to 23% as $\hat{T}$ increases from 30 to 120 months. In this case, larger estimation risk at long horizons is also offset by the better risk-return trade-off, so the optimal portfolio has a similar composition both at long and short horizons when parameter uncertainty is taken into account.

It is interesting to note that although Figure 2 shows that the consequences of changes in the initial value of *Trend* on the investment horizon effect are symmetric around its mean, the presence of stocks in the investment opportunity set reinforces the negative horizon effect (allocation to bonds drops by almost 16% in the horizons considered) and mitigates the positive horizon effect (allocation to bonds increases by 8%).

Finally, we consider changes in the dividend yield's initial value. This predictor is particularly interesting because its autocorrelation coefficient is close to 1 according

Figure 4

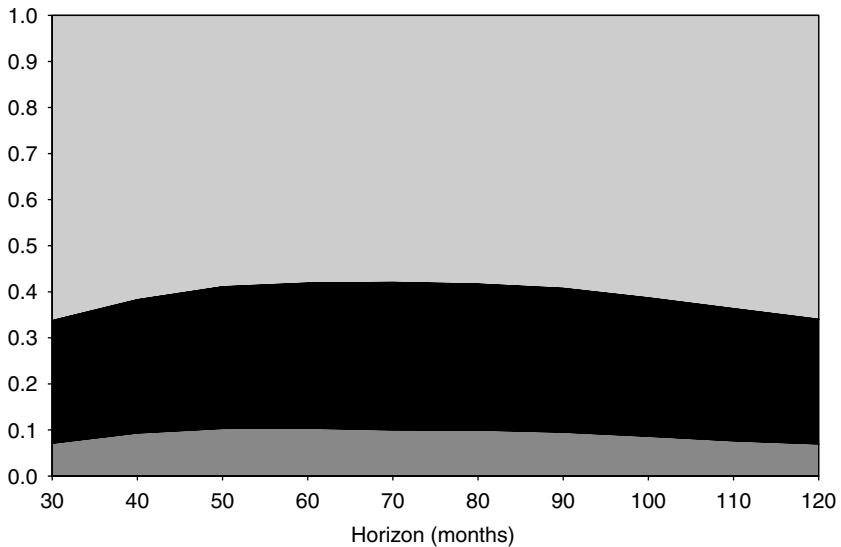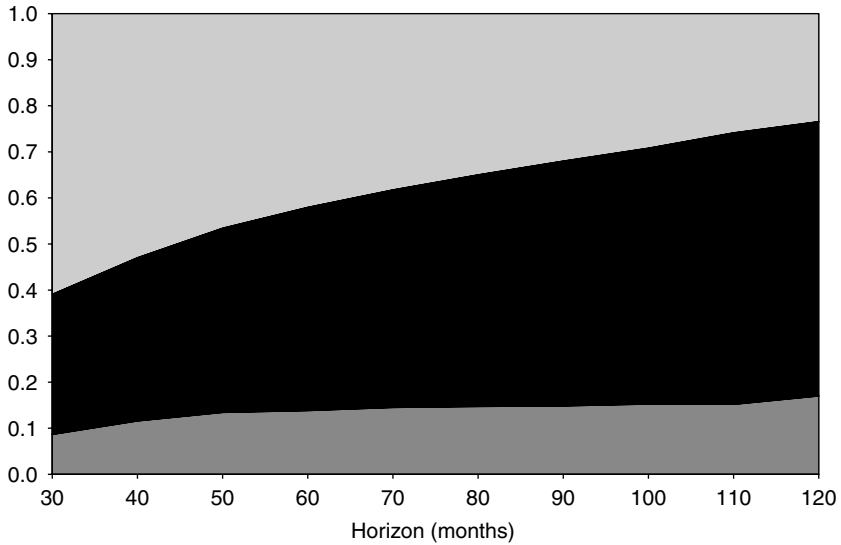Optimal Portfolio Choice as a Function of the Investment Horizon
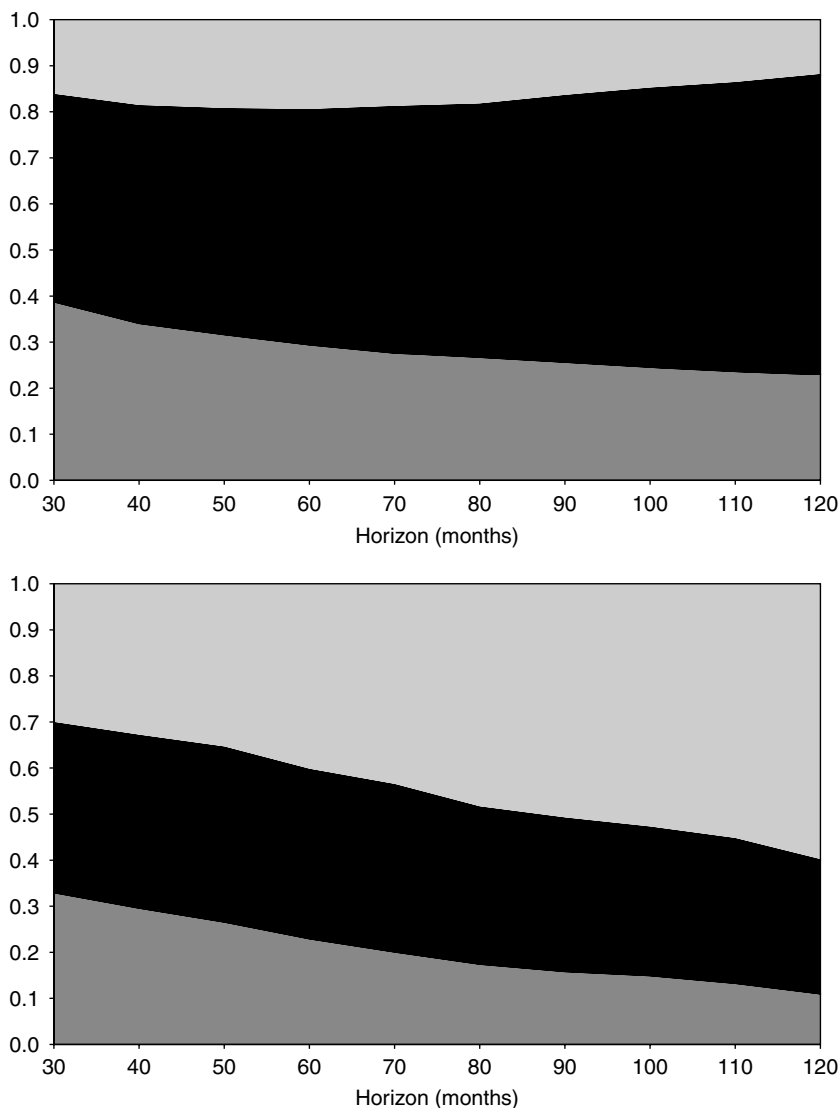
Notes:
The graph plots the static allocation to the riskless asset (light shade), 10-year bonds (medium shade) and stocks (dark shade) when returns are assumed to be generated by a VAR model with four predictors. The current value of predictor *Trend* (the trend of the stock index) equals -0.7, while the initial values of the rest of predictive variables are set at their sample means. In the lower graph, the investor takes parameter uncertainty into account. The risk aversion coefficient is 10.

to Table 2. This suggests that the investment horizon effects on the mean of cumulative returns are negligible due to lack of mean reversion in the predictor. Investment horizon effects in the optimal allocation to bonds can therefore only be

Figure 5

Optimal Portfolio Choice as a Function of the Investment Horizon
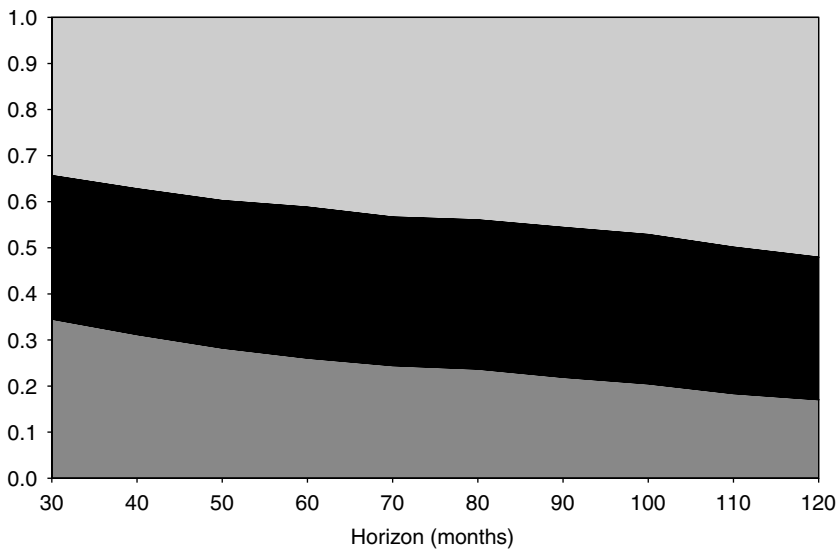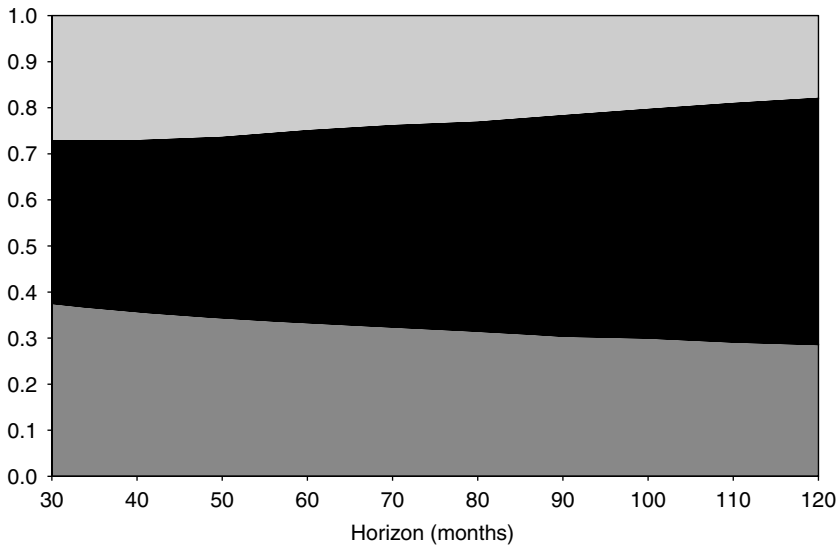

Notes:
The graph plots the static allocation to the riskless asset (light shade), 10-year bonds (medium shade) and stocks (dark shade) when returns are assumed to be generated by a VAR model with four predictors. The current value of predictor *Trend* (the trend of the stock index) equals 4.5, while the initial values of the rest of predictive variables are set at their sample means. In the lower graph, the investor takes parameter uncertainty into account. The risk aversion coefficient is 10.

negative and explained by either the lower riskiness of stocks as the horizon increases or by parameter uncertainty.

Figure 6 corresponds to the case when DP is initially at its 25th sample percentile. As expected, the presence of long term bonds in the risky portfolio decreases with

Figure 6

Optimal Portfolio Choice as a Function of the Investment Horizon
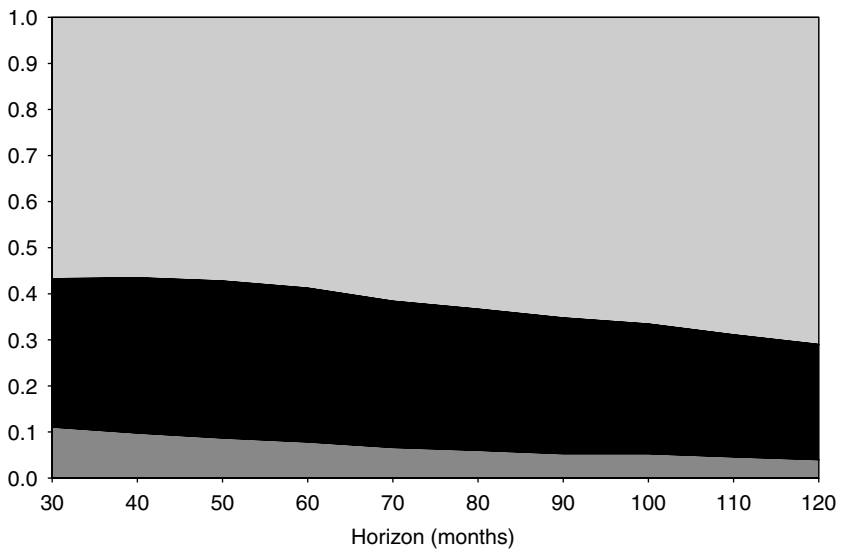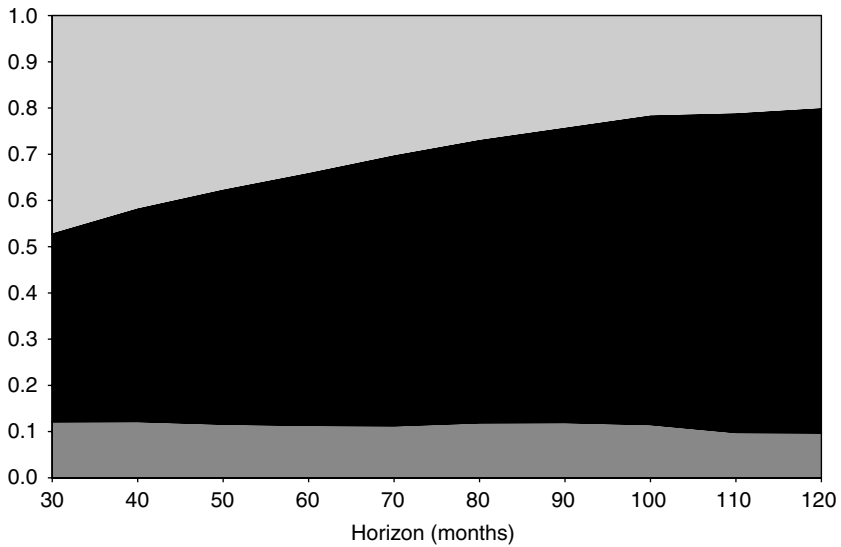

Notes:
The graph plots the static allocation to the riskless asset (light shade), 10-year bonds (medium shade) and stocks (dark shade) when returns are assumed to be generated by a VAR model with four predictors. The current value of predictor DP (dividend yield) equals 2.88, while the initial values of the rest of predictive variables are set at their sample means. In the lower graph, the investor takes parameter uncertainty into account. The risk aversion coefficient is 10.

the investment horizon at the same rate as in Figure 3, where all predictors were set at their initial values. Because investment in the risky asset portfolio increases, horizon effects in the optimal allocation to bonds are negative but not very

Figure 7

Optimal Portfolio Choice as a Function of the Investment Horizon

Notes:
The graph plots the static allocation to the riskless asset (light shade), 10-year bonds (medium shade) and stocks (dark shade) when returns are assumed to be generated by a VAR model with four predictors. The current value of predictor DP (dividend yield) equals 4.07, while the initial values of the rest of predictive variables are set at their sample means. In the lower graph, the investor takes parameter uncertainty into account. The risk aversion coefficient is 10.

pronounced if the investor ignores estimation risk. Finally, Figure 7 shows the results for an initial value of DP equal to its 75th sample percentile. Again, investment horizon effects on bond investment are negative. In this case, however, given

the predominance of stocks in the optimal portfolio, changes in the investor's allocation to long term bonds are very small in relative terms, and only become apparent when parameter uncertainty is considered explicitly.

4. CONCLUSIONS

This paper takes a deeper look into the consequences of return predictability on investment horizon effects in the optimal buy-and-hold asset allocation of a utility maximizing investor.

The paper's first contribution is a theoretical analysis of how the parameters in the model driving the return dynamics influence the evolution of the investor's opportunity set with the investment horizon. It is shown that under stationarity of the predictive variables, all investment horizons disappear in the long horizon limit. When a single risky asset is available for investment, it is demonstrated that changes in the variance of cumulative returns are not necessarily monotonic in the investment horizon. It is also shown that the investment horizon can have either a positive or a negative effect on the mean of cumulative returns. This is due to a diminishing influence of the predictor's effect on expected returns when the state variable is stationary. If this effect is positive and the predictor's initial value is above average, a long-term investor who believes in return predictability will expect a relatively lower return than a short term investor.

The paper also contributes to the literature on multi-period asset allocation by extending Barberis (2000) and Boudry and Gray (2003) to the portfolio choice problem when two risky assets – long-term US Government Bonds and US stocks – are available for investment both under parameter certainty and parameter uncertainty. The main conclusion is that under return predictability the risk-return characteristics of each risky asset can either improve or worsen with the investment horizon. The weight of each risky asset in the risky portfolio may therefore change as its relative appeal with respect to the other risky asset varies with the investment horizon.

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